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COLONNETTI'S MINIMUM PRINCIPLE EXTENSION TO GENERALLY NON-LINEAR MATERIALS

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Abstract In this paper two minimum principles are presented for the continuum problem with general non-linear materials (holonomic, non-holonomic, with hardening or softening, time-dependent, etc.). The proof of the first principle is based on the use of an *elastic auxiliary problem* associated to the original non-linear one and on the interpretation of the actual inelastic strains as unknown strains imposed on the elastic auxiliary solid. In a dual way the proof of the second principle is given through the imposition of suitable stresses on the elastic auxiliary solid.

Classical principles of elasticity and incremental elastoplasticity are then derived from the new principles as particular cases. Three simple illustrative examples are given.

1. INTRODUCTION

The continuum problem in the presence of generally non-linear material behaviour is often characterized by having no potential. This occurrence excludes the possibility of finding extremal formulations for it (in the classical sense) and, hence, rules out the relevant advantages. As is well known, among these advantages the following may be pointed out.

(1) The extremal formulations make the qualitative study of the problem easier, i.e. the study of the existence, uniqueness and regularity of the solution through the so-called direct methods of variational calculus (Dacorogna, 1989). Remarkable results in this direction have been obtained, for example, by Ball (1977a, b) in non-linear elasticity. The study of functionals with non-convex integrands is very attractive. In fact, such problems have, in general, no classical solution (i.e. in Sobolev space) and the arising of spatially chaotic structures is possible [see e.g. Dacorogna (1989); Schilling (1992)].

(2) The extremal formulations are particularly suitable for finding numerical solutions of the problem through direct solution procedures (based on the Ritz method or on the finite element approach) associated with optimization procedures (like the conjugate gradient method, etc.). The value of the functional during the climbing process may be used as a measure of the convergence and the value of the functional at the solution may be used to evaluate the approximation error. Besides, it is worth noting that the success of the finite element method in the elasticity problem (but more generally in elliptic problems) primarily depends on its variational structure.

In the present paper, through the generalization of Colonnetti's classical principle of elasticity in the presence of known distortions (Colonnetti, 1918, 1950), two general minimum principles are derived, which always ensure a variational formulation of the continuum problem in the presence of every inelastic (linear or non-linear) material behaviour.

As is well known, Colonnetti's approach regards the material non-linearities (creep, plasticity, etc.) as imposed strains in a supposedly linear elastic continuum. In the same way the two proposed formulations are derived envisaging the total non-linear strains as given by the superposition on the linear elastic continuum of unknown suitable distortions.

First, the non-linear material problem in the small displacement and strain range is formulated in Section 2. Then, in Section 3, the elastic auxiliary problem notion is introduced on which the generalization of Colonnetti's principle is based. In Section 4, a general non-linear material principle is derived (using a particular form of Colonnetti's principle) which generalizes the minimum complementary elastic energy principle. Analogously, in Section 5 a generalization of the minimum total potential elastic energy principle is found. A particular class of functionals for material complying with Drucker's stability principle is then presented in Section 6. Section 7 is devoted to the particularization of the two proposed principles to well-known classical principles of elasticity and incremental elastoplasticity. In Section 8 three simple illustrative examples are given in order to show the construction and use of the functionals in the case of a multiple solution or lack of solution and in the case of time-dependent constitutive law. In Section 9 a brief discussion is presented.

It is pointed out that the above principles have also been found in a parallel paper (Carini and De Donato, 1995) using a mathematical approach based on Tonti's (Tonti, 1984) general procedure of finding *extended* variational formulations of every non-linear problem. Finally, it is worth noting that the functionals obtained here present some analogies of the one suggested by Ortiz (1985) for the convection–diffusion problems.

2. PROBLEM FORMULATION AND COLONNETTI'S PRINCIPLES

2.1. *Problem formulation*

Consider a solid occupying a region Ω (in $\mathscr{R}^3$) with a smooth external surface Γ in a triaxial orthogonal Cartesian reference system. Γ_u and Γ_p (with $\Gamma = \Gamma_u \cup \Gamma_p$ and $\Gamma_u \cap \Gamma_p = \emptyset$) are the parts of the surface Γ where displacements and surface tractions are imposed, respectively, while $\mathbf{x} = (x_1, x_2, x_3)$ denotes the position vector of a material point in Ω.

The external actions on the solid, i.e. the volume forces $F_i(\mathbf{x}; t)$, the imposed (say thermal) strains $\theta_{ij}(\mathbf{x}; t)$ on Ω, the imposed displacements $v_i(\mathbf{x}; t)$ on Γ_u and the tractions $p_i(\mathbf{x}; t)$ on Γ_p, are given for any instant $t_0 \leqslant t \leqslant t_1$ of a known time interval $T = [t_0, t_1]$, through known time functions† (all external actions being vanishing for $t < t_0$).

Under the assumption of small strains and displacements ("geometric" linearity), the equilibrium and compatibility equations read (the index summation convention is adopted):

$$\sigma_{ij,j} + F_i = 0 \quad \text{in } \Omega \times \mathrm{T} \tag{1}$$

$$\sigma_{ij} n_j = p_i \quad \text{on } \Gamma_p \times \mathrm{T} \tag{2}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega \times \mathrm{T} \tag{3}$$

$$u_i = v_i \quad \text{on } \Gamma_u \times \mathrm{T}, \tag{4}$$

where σ_{ij} and ε_{ij} are components of the stress and the strain tensors $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, respectively; $(\cdot)_{,j} = \partial(\cdot)/\partial x_j$; n_j are the components of the outward normal—to the surface Γ—unit vector.

The assumed forms of the direct and inverse general non-linear constitutive law are, respectively,

$$\sigma_{ij} = \Psi_{ij}(\boldsymbol{\varepsilon} - \boldsymbol{\theta}) \tag{5}$$

$$\varepsilon_{ij} = \Phi_{ij}(\boldsymbol{\sigma}) + \theta_{ij} \tag{6}$$

or in a more symbolical way (without emphasizing the presence of the imposed distortions $\boldsymbol{\theta}$):

†In this context no inertia forces are considered; when it has no time meaning, the variable t is to be considered only as an event ordering parameter.

$$\sigma_{ij} = \Psi_{ij}(\boldsymbol{\varepsilon}) \tag{7}$$

$$\varepsilon_{ij} = \Phi_{ij}(\boldsymbol{\sigma}), \tag{8}$$

where Ψ_{ij} and Φ_{ij} are generally non-linear operators.

With the above notation any known non-linear material behaviour (history dependent or not) may be considered (like viscoelastic, elastoplastic with hardening or softening, viscoplastic, etc.). We assume, for simplicity, that the solid is originally undisturbed in the sense of the initial conditions, given by:

$$u_i = \varepsilon_{ij} = \sigma_{ij} = 0 \quad \text{on}\, \bar{\Omega} \times (-\infty, t_0), \tag{9}$$

where $\bar{\Omega} = \Omega \cup \Gamma$. The following additional assumptions are made.

(1) The constitutive law is invertible, i.e.

$$\Phi_{ij}(\cdot) = \Psi_{ij}^{-1}(\cdot). \tag{10}$$

(2) The direct and inverse form of the constitutive law may be expressed as follows:

$$\sigma_{ij}(\mathbf{x}\,;t) = D_{ijhk}(\mathbf{x})\varepsilon_{hk}(\mathbf{x}\,;t) + \Psi_{ij}^{n}(\boldsymbol{\varepsilon},\boldsymbol{\theta}) \tag{11}$$

$$\varepsilon_{ij}(\mathbf{x}\,;t) = B_{ijhk}(\mathbf{x})\sigma_{hk}(\mathbf{x}\,;t) + \Phi_{ij}^{n}(\boldsymbol{\sigma},\boldsymbol{\theta}). \tag{12}$$

In the following we will refer to this form of the constitutive law, without emphasizing the inelastic part dependence from the imposed distorsions $\boldsymbol{\theta}$. In eqns (11) and (12), $D_{ijhk} = B_{ijhk}^{-1}$ is the elastic modulus tensor which is assumed to have the following usual symmetry properties:

$$D_{ijhk} = D_{jihk} = D_{hkij} \tag{13}$$

and the property of positive definiteness:

$$D_{ijhk}(\mathbf{x})\gamma_{ij}\gamma_{hk} > 0 \tag{14}$$

for every $\mathbf{x} \in \Omega$ and for every non-vanishing double symmetric tensor $\boldsymbol{\gamma}$. Assumption 1 will ensure the deduction of both the minimum principles (Sections 4 and 5), while its relaxation allows only one of the two quoted principles to be derived depending on the available form (direct or inverse) of the constitutive law.

In the following the problem defined by eqns (1)–(9) will be referred as problem P. Examples of the splitting of the constitutive law into a linear elastic and inelastic (linear or non-linear) part as indicated in eqns (11) and (12) are given below.

(1) Linear viscoelastic law [see e.g. Christensen (1982)]:

$$\sigma_{ij}(\mathbf{x}\,;t) = H_{ijhk}(\mathbf{x}\,;0)[\varepsilon_{hk}(\mathbf{x}\,;t) - \theta_{hk}(\mathbf{x}\,;t)] + \int_0^t \frac{\partial H_{ijhk}(\mathbf{x}\,;t-\tau)}{\partial(t-\tau)}[\varepsilon_{hk}(\mathbf{x}\,;\tau) - \theta_{hk}(\mathbf{x}\,;\tau)]\,\mathrm{d}\tau, \tag{15}$$

where H_{ijhk} is the relaxation viscous kernel. In this case

$$D_{ijhk}(\mathbf{x}) \equiv H_{ijhk}(\mathbf{x}\,;0) \tag{16}$$

represents the elastic istantaneous modulus tensor, while

$$\Psi^{v}_{ij}(\varepsilon(\mathbf{x};t)) \equiv -H_{ijhk}(\mathbf{x};0)\theta_{hk}(\mathbf{x};t)+\int_0^t \frac{\partial H_{ijhk}(\mathbf{x};t-\tau)}{\partial(t-\tau)}[\varepsilon_{hk}(\mathbf{x};\tau)-\theta_{hk}(\mathbf{x};\tau)]\,\mathrm{d}\tau \quad (17)$$

represents the inelastic viscous part of the constitutive law. The above form of the viscoelastic linear law was used extensively by Carini *et al.* (1995).

(2) Elastoplastic constitutive law [see e.g. Halphen and Nguyen (1975)]:

$$\sigma_{ij} = D_{ijhk}e_{hk}, \quad \varepsilon_{ij} = e_{ij}+\varepsilon^{p}_{ij}+\theta_{ij} \quad (18)$$

$$\phi(\sigma_{ij}, q_h) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \phi\dot{\lambda} = 0 \quad (19)$$

$$\dot{\varepsilon}^{p}_{ij} = \frac{\partial\psi}{\partial\sigma_{ij}}(\sigma_{ij}, q_h)\dot{\lambda}, \quad \dot{\eta}_h = -\frac{\partial\psi}{\partial q_h}(\sigma_{ij}, q_k)\dot{\lambda} \quad (20)$$

$$q_h = \frac{\partial W}{\partial \eta_h}(\eta_k), \quad (21)$$

where eqns (18) reflect Hooke's law and strain additivity (ε^{p}_{ij} denote plastic strains and a superimposed point of the symbol means infinitesimal increment with respect to t). The first of relations (19) defines the yield function and the elastic domain, while the second two express the loading unloading criterion: eqns (20) express the generalized flow rule of the plastic material model (associative if $\psi = \phi$). We denote by q_h ($h = 1, \ldots, n_s$) the static internal variables and by η_h the conjugate kinematic internal variables. In eqn (21), which relates static to kinematic internal variables, W can be interpreted as the *stored* free energy due to structural rearrangements at the microscale. In this case

$$\Psi^{v}_{ij} = -D_{ijhk}(\varepsilon^{p}_{hk}+\theta_{hk}). \quad (22)$$

Here ε^{p}_{ij} depend non-linearly on ε_{ij} through eqns (18)–(21).

(3) Other examples of constitutive law additivity of type (11) and (12) are given by the viscoelastoplastic [see e.g. Rabotnov (1969)] and thermoelastic cases. The additivity of elastic and inelastic strains for many classical constitutive laws as seen in the above examples is well known. There are, however, constitutive laws for which the mentioned additivity is no longer valid, such as, for instance, the elastoplastic constitutive laws with damage.

(4) In any case it is worth noting that the decomposition (11) and (12) is always (at least formally) possible even if it may not be so easy to understand the physical meaning of the reversible elastic and of the dissipative inelastic parts.

2.2. *Colonnetti's principle*

Castigliano (1879) and later Colonnetti (1918, 1950) dealt with the problem of finding the stress field in an elastic continuum under given distortions. In particular, the so-called Colonnetti's principle states that: the stress field σ_{ij} solution of the elastic problem under given external loads F_i in Ω, p_i on Γ_p, imposed displacements v_i on Γ_u and imposed distortions θ_{ij} in Ω, minimizes the following functional:

$$F_c[\sigma^*_{ij}] = \frac{1}{2}\int_\Omega \sigma^*_{ij}B_{ijhk}\sigma^*_{hk}\,\mathrm{d}\Omega - \int_{\Gamma_u}\sigma^*_{ij}n_j v_i\,\mathrm{d}\Gamma + \int_\Omega \sigma^*_{ij}\theta_{ij}\,\mathrm{d}\Omega \quad (23)$$

under the conditions:

$$\sigma^*_{ij,j}+F_i = 0 \quad \text{in} \quad \Omega \quad (24)$$

$$\sigma^*_{ij}n_j = p_i \quad \text{on} \quad \Gamma_p. \quad (25)$$

However, in the presence of non-linear material behaviour, the distortions due to the non-linearity of the material are *a priori* unknown and are part of the problem solution; Colonnetti's principle, in the above form, is not suitable for direct use and it may be useful to find a different form of it. To this aim it may be more advantageous to use Colonnetti's functional [eqn (23)] in the following equivalent form which may be found using the principle of virtual work:

$$\tilde{F}_{ce}[\sigma^*_{ij}] = \frac{1}{2}\int_\Omega \tilde{\sigma}^d_{ij} B_{ijhk} \tilde{\sigma}^d_{hk}\, d\Omega, \tag{26}$$

where $\tilde{\sigma}^d_{ij}$ is the solution of the elastic problem for the following imposed strains:

$$\tilde{\theta}_{ij} = -B_{ijhk}\sigma^*_{hk} - \theta_{ij}. \tag{27}$$

Obviously the minimum of $\tilde{F}_{ce}$ is reached when $\tilde{\theta}_{ij}$ is compatible, i.e. a compatible displacement field u^o_i may be found such that:

$$\tfrac{1}{2}(u^o_{i,j} + u^o_{j,i}) = B_{ijhk}\sigma^*_{hk} + \theta_{ij} \quad \text{in}\,\Omega \tag{28}$$

$$u^o_i = v_i \quad \text{on}\,\Gamma_u \tag{29}$$

and the value of $\tilde{F}_{ce}$ vanishes since $\tilde{\sigma}^d_{ij}$ does. In this way Colonnetti's functional, in the form (26), may be interpreted as the elastic energy due to distorsions $\tilde{\theta}_{ij}$ in Ω given by eqn (27).

The above form [eqn (26)] of Colonnetti's functional is particularly suitable for its extension to the case of stress-dependent and stress-history-dependent distortions [i.e. $\theta_{ij} = \theta_{ij}(\boldsymbol{\sigma})$] due to the non-linearity of the material behaviour as in the case for plastic yielding.

The new principles of next sections will be based on the above form [eqn (26)].

3. THE ELASTIC AUXILIARY PROBLEM NOTION

3.1. *Statically admissible stress and kinematically admissible strain fields*

A stress distribution field $\sigma^*_{ij}(\mathbf{x}, t)$ is defined as *statically admissible* when it satisfies the following equilibrium equations:

$$\sigma^*_{ij,j} + F_i = 0 \quad \text{in}\,\Omega\times\mathrm{T} \tag{30}$$

$$\sigma^*_{ij}n_j = p_i \quad \text{on}\,\Gamma_p\times\mathrm{T}. \tag{31}$$

The corresponding (generally non-compatible) deformations derived through the constititive law [eqn (8)] will be referred to as

$$\varepsilon^*_{ij} = \Phi_{ij}(\boldsymbol{\sigma}^*). \tag{32}$$

Analogously a strain field $\varepsilon^o_{ij}(\mathbf{x}, t)$ is defined as *kinematically admissible* if it can be derived by means of the equation:

$$\varepsilon^o_{ij} = \tfrac{1}{2}(u^o_{i,j} + u^o_{j,i}) \quad \text{in}\,\Omega\times\mathrm{T} \tag{33}$$

from a displacement field $u^o_i(\mathbf{x}, t)$ satisfying the boundary conditions

$$u_i^o = r_i \quad \text{on } \Gamma_u \times \mathrm{T}. \tag{34}$$

The corresponding (generally non-equilibrated) stress field derived through the constitutive law [eqn (7)] will be referred to as

$$\sigma_{ij}^o = \Psi_{,ij}(\boldsymbol{\varepsilon}^o). \tag{35}$$

3.2. *The imposed strains elastic auxiliary problem* P^d

The notion of elastic auxiliary problem derives simply from the interpretation of the constitutive law of the original problem P as a sum of two parts, the first of which corresponds to the linear elastic behaviour.

When the inverse constitutive law [eqn (12)] is considered, the imposed strain elastic auxiliary problem P^d is defined by the original solid with the elastic material properties corresponding to B_{ijhk} in the presence of vanishing external loads, homogeneous boundary conditions and imposed strains d_{ij} given by

$$d_{ij} = -\Phi_{ij}^n(\boldsymbol{\sigma}^*), \tag{36}$$

$\boldsymbol{\sigma}^*$ being any statically admissible stress field.

The imposed strain elastic auxiliary problem P^d is then defined by the following governing equations:

$$\text{Problem } P^d: \quad \begin{aligned} &\sigma_{ij,j}^d = 0 && \text{in } \Omega \times \mathrm{T} && (37)\\ &\sigma_{ij}^d n_j = 0 && \text{on } \Gamma_p \times \mathrm{T} && (38)\\ &\varepsilon_{ij}^d = \tfrac{1}{2}(u_{i,j}^d + u_{j,i}^d) && \text{in } \Omega \times \mathrm{T} && (39)\\ &u_i^d = 0 && \text{on } \Gamma_u \times \mathrm{T} && (40)\\ &\varepsilon_{ij}^d = B_{ijhk}\sigma_{hk}^d + d_{ij}. && && (41) \end{aligned}$$

Two other related imposed strain elastic auxiliary problems $\hat{P}^d$, $\tilde{P}^d$ will be considered in the following depending on the kind of imposed strains and on the presence of external loads, i.e.

$$\text{Problem } \hat{P}^d: \quad \begin{aligned} &\hat{\sigma}_{ij,j}^d + F_i = 0 && \text{in } \Omega \times \mathrm{T} && (42)\\ &\hat{\sigma}_{ij}^d n_j = p_i && \text{on } \Gamma_p \times \mathrm{T} && (43)\\ &\hat{\varepsilon}_{ij}^d + \eta_{ij} = \tfrac{1}{2}(\hat{u}_{i,j}^d + \hat{u}_{j,i}^d) && \text{in } \Omega \times \mathrm{T} && (44)\\ &\hat{u}_i^d = r_i && \text{on } \Gamma_u \times \mathrm{T} && (45)\\ &\hat{\varepsilon}_{ij}^d = B_{ijhk}\hat{\sigma}_{hk}^d. && && (46) \end{aligned}$$

$$\text{Problem } \tilde{P}^d: \quad \begin{aligned} &\tilde{\sigma}_{ij,j}^d = 0 && \text{in } \Omega \times \mathrm{T} && (47)\\ &\tilde{\sigma}_{ij}^d n_j = 0 && \text{on } \Gamma_p \times \mathrm{T} && (48)\\ &\tilde{\varepsilon}_{ij}^d + \eta_{ij} = \tfrac{1}{2}(\tilde{u}_{i,j}^d + \tilde{u}_{j,i}^d) && \text{in } \Omega \times \mathrm{T} && (49)\\ &\tilde{u}_i^d = 0 && \text{on } \Gamma_u \times \mathrm{T} && (50)\\ &\tilde{\varepsilon}_{ij}^d = B_{ijhk}\tilde{\sigma}_{hk}^d, && && (51) \end{aligned}$$

where

$$\eta_{ij} = \varepsilon_{ij}^o + d_{ij} - B_{ijhk}\sigma_{hk}^* = \varepsilon_{ij}^o - \Phi_{ij}(\boldsymbol{\sigma}^*) \tag{52}$$

and where ε^o_{ij} is any kinematically admissible strain field. In other words $\hat{u}^d_i, \hat{\varepsilon}^d_{ij}, \hat{\sigma}^d_{ij}$ of the problem $\hat{P}^d$ represent the elastic response to the external loads F_i, p_i, and to the imposed strains η_{ij} and displacements v_i on Γ_u. In a similar way, $\tilde{u}^d_i, \tilde{\varepsilon}^d_{ij}, \tilde{\sigma}^d_{ij}$ of the problem $\tilde{P}^d$ refer only to the case of imposed strains η_{ij}, and $\tilde{\sigma}^d_{ij}$ is independent from the choice of the kinematically strain field ε^o_{ij}. Obvious relations between the stress, strain and displacements of all the above imposed strain elastic auxiliary problems P^d, $\hat{P}^d$, $\tilde{P}^d$ are:

$$\hat{u}^d_i = u^*_i + u^d_i; \tag{53a}$$

$$\hat{\varepsilon}^d_{ij} = \varepsilon^*_{ij} + \varepsilon^d_{ij}; \tag{53b}$$

$$\hat{\sigma}^d_{ij} = \sigma^*_{ij} + \sigma^d_{ij} \tag{53c}$$

$$\tilde{u}^d_i = \hat{u}^d_i - u^{el}_i; \tag{54a}$$

$$\tilde{\varepsilon}^d_{ij} = \hat{\varepsilon}^d_{ij} - \varepsilon^{el}_{ij}; \tag{54b}$$

$$\tilde{\sigma}^d_{ij} = \hat{\sigma}^d_{ij} - \sigma^{el}_{ij}, \tag{54c}$$

where $u^{el}_i, \varepsilon^{el}_{ij}, \sigma^{el}_{ij}$ represent the known response of the solid under the external actions F_i, p_i, v_i and under the assumption of linear elastic behaviour with material properties corresponding to B_{ijhk}.

3.3. *The imposed stress elastic auxiliary problem* P^s

When the direct constitutive law [eqn (11)] is considered, the imposed stress elastic auxiliary problem P^s is defined by the original solid with the elastic material properties corresponding to D_{ijhk} in the presence of vanishing external loads, homogeneous boundary conditions and imposed stresses s_{ij} given by

$$s_{ij} = -\Psi^n_{,ij}(\varepsilon^o). \tag{55}$$

ε^o being any kinematically admissible strain field.

The imposed stress elastic auxiliary problem P^s is then defined by the following governing equations:

Problem P^s:

$$\sigma^s_{ij,j} = 0 \quad \text{in}\, \Omega \times \mathrm{T} \tag{56}$$

$$\sigma^s_{ij} n_j = 0 \quad \text{on}\, \Gamma_p \times \mathrm{T} \tag{57}$$

$$\varepsilon^s_{ij} = \tfrac{1}{2}(u^s_{i,j} + u^s_{j,i}) \quad \text{in}\, \Omega \times \mathrm{T} \tag{58}$$

$$u^s_i = 0 \quad \text{on}\, \Gamma_u \times \mathrm{T} \tag{59}$$

$$\sigma^s_{ij} = D_{ijhk}\varepsilon^s_{hk} + s_{ij}. \tag{60}$$

Two other related imposed stress elastic auxiliary problems $\hat{P}^s$, $\tilde{P}^s$ will be considered in the following depending on the kind of imposed stresses and on the presence of external loads, i.e.

Problem $\hat{P}^s$:

$$\hat{\sigma}^s_{ij,j} + R_i = 0 \quad \text{in}\, \Omega \times \mathrm{T} \tag{61}$$

$$\hat{\sigma}^s_{ij} n_j = g_i \quad \text{on}\, \Gamma_p \times \mathrm{T} \tag{62}$$

$$\hat{\varepsilon}^s_{ij} = \tfrac{1}{2}(\hat{u}^s_{i,j} + \hat{u}^s_{j,i}) \quad \text{in}\, \Omega \times \mathrm{T} \tag{63}$$

$$\hat{u}^s_i = v_i \quad \text{on}\, \Gamma_u \times \mathrm{T} \tag{64}$$

$$\hat{\sigma}^s_{ij} = D_{ijhk}\hat{\varepsilon}^s_{hk} \tag{65}$$

$$\text{Problem } \tilde{P}^s: \quad \begin{aligned} &\tilde{\sigma}^s_{ij,j} + R_i - F_i = 0 && \text{in } \Omega \times \mathrm{T} && (66)\\ &\tilde{\sigma}^s_{ij} n_j = g_i - p_i && \text{on } \Gamma_p \times \mathrm{T} && (67)\\ &\tilde{\varepsilon}^s_{ij} = \tfrac{1}{2}(\tilde{u}^s_{i,j} + \tilde{u}^s_{j,i}) && \text{in } \Omega \times \mathrm{T} && (68)\\ &\tilde{u}^s_i = 0 && \text{on } \Gamma_u \times \mathrm{T} && (69)\\ &\tilde{\sigma}^s_{ij} = D_{ijhk}\tilde{\varepsilon}^s_{hk}, && && (70) \end{aligned}$$

where

$$R_i = -[\Psi_{ij}(\boldsymbol{\varepsilon}^o)]_{,j} \tag{71}$$

$$g_i = [\Psi_{ij}(\boldsymbol{\varepsilon}^o)]n_j. \tag{72}$$

In other words $\hat{u}^s_i, \hat{\varepsilon}^s_{ij}, \hat{\sigma}^s_{ij}$ of the problem $\hat{P}^s$ represent the elastic response to the external loads R_i, g_i and to the imposed displacements v_i on Γ_u. In a similar way, $\tilde{u}^s_i, \tilde{\varepsilon}^s_{ij}, \tilde{\sigma}^s_{ij}$ of the problem $\tilde{P}^s$ refer to the case of external loads $R_i - F_i$ and $q_i - p_i$. Obvious relations between the stress, strain and displacements of all the above imposed stress elastic auxiliary problems P^s, $\hat{P}^s$, $\tilde{P}^s$ are:

$$\hat{u}^s_i = u^o_i + u^s_i; \tag{73a}$$

$$\hat{\varepsilon}^s_{ij} = \varepsilon^o_{ij} + \varepsilon^s_{ij}; \tag{73b}$$

$$\hat{\sigma}^s_{ij} = \sigma^o_{ij} + \sigma^s_{ij} \tag{73c}$$

$$\tilde{u}^s_i = \hat{u}^s_i - u^{ef}_i; \tag{74a}$$

$$\tilde{\varepsilon}^s_{ij} = \hat{\varepsilon}^s_{ij} - \varepsilon^{ef}_{ij}; \tag{74b}$$

$$\tilde{\sigma}^s_{ij} = \hat{\sigma}^s_{ij} - \sigma^{ef}_{ij}. \tag{74c}$$

Remarks. (1) The solutions $u^d_i, \varepsilon^d_{ij}, \sigma^d_{ij}$ and $u^s_i, \varepsilon^s_{ij}, \sigma^s_{ij}$ of the elastic auxiliary problem P^d and P^s, respectively, can be represented through the so-called Green functions relative to the considered elastic auxiliary solid. Let $\mathbf{G}^{lm}(\mathbf{x}, \boldsymbol{\xi})$ be the influence matrix-valued functions (Green functions) describing effects in $\mathbf{x} \in \Omega$, due to specified singularities applied in $\boldsymbol{\xi} \in \Omega$; the "effect" is specified by the first superscript, and the "cause" by the second one. For instance $u^d_i, \varepsilon^d_{ij}, \sigma^d_{ij}$ can be represented thus:

$$u^d_i(\mathbf{x}) = \int_\Omega G^{u\theta}_{i\alpha\beta}(\mathbf{x}, \boldsymbol{\xi}) d_{\alpha\beta}(\boldsymbol{\xi})\, \mathrm{d}\Omega_\xi \tag{75}$$

$$\varepsilon^d_{ij}(\mathbf{x}) = \int_\Omega G^{\varepsilon\theta}_{ij\alpha\beta}(\mathbf{x}, \boldsymbol{\xi}) d_{\alpha\beta}(\boldsymbol{\xi})\, \mathrm{d}\Omega_\xi \tag{76}$$

$$\sigma^d_{ij}(\mathbf{x}) = \int_\Omega G^{\sigma\theta}_{ij\alpha\beta}(\mathbf{x}, \boldsymbol{\xi}) d_{\alpha\beta}(\boldsymbol{\xi})\, \mathrm{d}\Omega_\xi \tag{77}$$

with the obvious relations between the Green functions

$$G^{\varepsilon\theta}_{ij\alpha\beta} = \frac{1}{2}\left(\frac{\partial G^{u\theta}_{i\alpha\beta}}{\partial x_j} + \frac{\partial G^{u\theta}_{j\alpha\beta}}{\partial x_i}\right); \quad G^{\sigma\theta}_{ij\alpha\beta} = D_{ijhk}G^{\varepsilon\theta}_{hk\alpha\beta}. \tag{78}$$

(2) The solutions of the elastic auxiliary problems P^d and P^s coincide in the case of $d_{ij} = -\Phi^n_{ij}(\boldsymbol{\sigma})$ [i.e. taking $\boldsymbol{\sigma}^* \equiv \boldsymbol{\sigma}$ in eqn (36)] and $s_{ij} = -\Psi^n_{ij}(\boldsymbol{\varepsilon})$ [i.e. taking $\boldsymbol{\varepsilon}^o \equiv \boldsymbol{\varepsilon}$ in eqn (55)].

4. AN EXTENDED COMPLEMENTARY ENERGY PRINCIPLE

Let us consider the following functional of a statically admissible stress field history $\sigma^*_{ij}(\mathbf{x}; t)$:

$$F^{\mathrm{I}}_{ce}[\sigma^*_{ij}] = \int_T \left\{\frac{1}{2}\int_\Omega \sigma^*_{ij}B_{ijhk}\sigma^*_{hk}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \sigma^*_{ij}\,\mathrm{d}\Gamma + \int_\Omega \Phi^n_{ij}(\boldsymbol{\sigma}^*)(\sigma^*_{ij} - \sigma^{el}_{ij})\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \sigma^d_{ij}B_{ijhk}\sigma^d_{hk}\,\mathrm{d}\Omega\right\}\mathrm{d}t, \tag{79}$$

where σ^d_{ij} are the stresses of the imposed strain elastic auxiliary problem P^d in the presence of vanishing external load, of homogeneous boundary conditions and of imposed strains $d_{ij} = -\Phi^n_{ij}(\boldsymbol{\sigma}^*)$ and σ^{el}_{ij} are the elastic stresses of the solid under the original external actions F_i, p_i, v_i with the assumption of linear elastic behaviour with material properties corresponding to B_{ijhk}.

Proposition 1

The functional (79) is equivalent to each of the following four functionals:

$$F^{\mathrm{II}}_{ce}[\sigma^*_{ij}] = \int_T \left\{\frac{1}{2}\int_\Omega \sigma^*_{ij}B_{ijhk}\sigma^*_{hk}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \sigma^*_{ij}\,\mathrm{d}\Gamma + \int_\Omega \Phi^n_{ij}(\boldsymbol{\sigma}^*)\sigma^*_{ij}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \sigma^d_{ij}B_{ijhk}\sigma^d_{hk}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \sigma^d_{ij}\,\mathrm{d}\Gamma + \int_\Omega F_i u^d_i\,\mathrm{d}\Omega + \int_{\Gamma_p} p_i u^d_i\,\mathrm{d}\Gamma\right\}\mathrm{d}t, \tag{80}$$

$$F^{\mathrm{III}}_{ce}[\sigma^*_{ij}] = \int_T \left\{\frac{1}{2}\int_\Omega \Phi_{ij}(\boldsymbol{\sigma}^*)\hat{\sigma}^d_{ij}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \hat{\sigma}^d_{ij}\,\mathrm{d}\Gamma + \frac{1}{2}\int_\Omega F_i u^d_i\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Gamma_p} p_i u^d_i\,\mathrm{d}\Gamma\right\}\mathrm{d}t, \tag{81}$$

$$F^{\mathrm{IV}}_{ce}[\sigma^*_{ij}] = \int_T \left\{\frac{1}{2}\int_\Omega \hat{\sigma}^d_{ij}B_{ijhk}\hat{\sigma}^d_{hk}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \hat{\sigma}^d_{ij}\,\mathrm{d}\Gamma\right\}\mathrm{d}t, \tag{82}$$

$$F^{\mathrm{V}}_{ce}[\sigma^*_{ij}] = \int_T \left\{\frac{1}{2}\int_\Omega \tilde{\sigma}^d_{ij}B_{ijhk}\tilde{\sigma}^d_{hk}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \sigma^{el}_{ij}\varepsilon^{el}_{ij}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \sigma^{el}_{ij}\,\mathrm{d}\Gamma\right\}\mathrm{d}t. \tag{83}$$

Proof. The functional (80) is derived from functional (79) using eqn (A1) of the Appendix. The functional (81) is derived from functional (80) using eqns (A2), (A3) and (53c). The functional (82) is derived from functional (81) using eqns (32), (46), (53b) and (A4). The functional (83) is derived from the substitution of eqn (A5) in the functional (82) and using eqns (46), (51) and (54b,c) ■

Proposition 2

A statically admissible stress field σ_{ij}^* is a (or the) solution of the problem P [eqns (1)–(9)] if and only if it minimizes (absolute minimum) the functional (79) [or any of the equivalent functionals (80)–(83)]. The problem P has at least one solution if and only if the functional (79) assumes, at the minimum, the following value:

$$F_{ce}^{0} = \int_T \left\{ \frac{1}{2} \int_\Omega \sigma_{ij}^{ef} \varepsilon_{ij}^{ef} \, \mathrm{d}\Omega - \int_{\Gamma_v} v_i n_j \sigma_{ij}^{ef} \, \mathrm{d}\Gamma \right\} \mathrm{d}t. \tag{84}$$

Proof. The proof is given using the functional $F_{ce}^{V}[\sigma_{ij}^*]$ showing that the difference $F_{ce}^{V} - F_{ce}^{0}$ is always non-negative, i.e.

$$F_{ce}^{V} - F_{ce}^{0} = \frac{1}{2} \int_T \int_\Omega \tilde{\varepsilon}_{ij}^{d} \tilde{\sigma}_{ij}^{d} \, \mathrm{d}\Omega \, \mathrm{d}t \geqslant 0. \tag{85}$$

This can be easily recognized because the second member of eqn (85) represents the elastic deformation energy of the solid due to the imposed strains $\eta_{ij} = \varepsilon_{ij}^{o} - \Phi_{ij}(\boldsymbol{\sigma}^*)$.

The functional F_{ce}^{V} attains its minimum value F_{ce}^{0} if and only if there exists a kinematically admissible strain field ε_{ij}^{o} and a statically admissible stress field σ_{ij}^* such that $\eta_{ij} = 0$, i.e. if and only if

$$\varepsilon_{ij}^{o} = \Phi_{ij}(\boldsymbol{\sigma}^*) \quad \text{in } \Omega \times \mathrm{T} \tag{86}$$

which represents the compatibility equations and the constitutive law. Equation (86) together with the conditions (30) and (31) represent the whole of the governing equations of the original problem P [eqns (1)–(9)]. If at the minimum

$$F_{ce}^{V} > F_{ce}^{0} \tag{87}$$

then no stress field σ_{ij}^* [satisfying restrictions (30) and (31)] exists such as to satisfy also equation (86) at the same time. Then the original problem P has no solution. ■

5. AN EXTENDED TOTAL POTENTIAL ENERGY PRINCIPLE

Let us consider the following functional of any kinematically admissible displacement field history $u_i^o(\mathbf{x}; t)$:

$$F_{tpe}^{1}[u_i^o] = \int_T \left\{ \frac{1}{2} \int_\Omega \varepsilon_{ij}^{o} D_{ijhk} \varepsilon_{hk}^{o} \, \mathrm{d}\Omega - \int_\Omega F_i u_i^o \, \mathrm{d}\Omega - \int_{\Gamma_p} p_i u_i^o \, \mathrm{d}\Gamma \right.$$
$$\left. + \int_\Omega \Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{o})(\varepsilon_{ij}^{o} - \varepsilon_{ij}^{ef}) \, \mathrm{d}\Omega + \frac{1}{2} \int_\Omega \varepsilon_{ij}^{s} D_{ijhk} \varepsilon_{hk}^{s} \, \mathrm{d}\Omega \right\} \mathrm{d}t, \tag{88}$$

where ε_{ij}^{s} are the strains of the imposed stress elastic auxiliary problem P^{s} in the presence of vanishing external load, omogeneous boundary conditions and imposed stresses $s_{ij} = -\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{o})$ and ε_{ij}^{ef} are the elastic strains of the solid under the original external actions F_i, p_i, v_i with the assumption of linear elastic behaviour with material properties corresponding to D_{ijhk}.

Proposition 3

The functional (88) is equivalent to each of the following four functionals:

$$F^{\mathrm{II}}_{tpe}[u^o_i] = \int_T \left\{ \frac{1}{2}\int_\Omega \varepsilon^o_{ij} D_{ijhk} \varepsilon^o_{hk}\,\mathrm{d}\Omega - \int_\Omega F_i u^o_i\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i u^o_i\,\mathrm{d}\Gamma \right.$$

$$\left. + \int_\Omega \Psi^n_{ij}(\boldsymbol{\varepsilon}^o)\varepsilon^o_{ij}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \varepsilon^s_{ij} D_{ijhk}\varepsilon^s_{hk}\,\mathrm{d}\Omega + \int_{\Gamma_u} v_i n_j \sigma^s_{ij}\,\mathrm{d}\Gamma - \int_\Omega F_i u^s_i\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i u^s_i\,\mathrm{d}\Gamma \right\}\mathrm{d}t \tag{89}$$

$$F^{\mathrm{III}}_{tpe}[u^o_i] = \int_T \left\{ \frac{1}{2}\int_\Omega \Psi_{ij}(\boldsymbol{\varepsilon}^o)\hat{\varepsilon}^s_{ij}\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Gamma_u} v_i n_j \sigma^s_{ij}\,\mathrm{d}\Gamma - \int_\Omega F_i \hat{u}^s_i\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i \hat{u}^s_i\,\mathrm{d}\Gamma \right\}\mathrm{d}t \tag{90}$$

$$F^{\mathrm{IV}}_{tpe}[u^o_i] = \int_T \left\{ \frac{1}{2}\int_\Omega \hat{\varepsilon}^s_{ij} D_{ijhk}\hat{\varepsilon}^s_{hk}\,\mathrm{d}\Omega - \int_\Omega F_i \hat{u}^s_i\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i \hat{u}^s_i\,\mathrm{d}\Gamma \right\}\mathrm{d}t \tag{91}$$

$$F^{\mathrm{V}}_{tpe}[u^o_i] = \int_T \left\{ \frac{1}{2}\int_\Omega \tilde{\sigma}^s_{ij}\tilde{\varepsilon}^s_{ij}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \sigma^{ef}_{ij}\varepsilon^{ef}_{ij}\,\mathrm{d}\Omega - \int_\Omega F_i u^{ef}_i\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i u^{ef}_i\,\mathrm{d}\Gamma \right\}\mathrm{d}t. \tag{92}$$

Proof. The functional (89) is derived from functional (88) using eqn (A6). The functional (90) is derived from functional (89) using eqns (A7), (A8) and (73a,b). The functional (91) is derived from functional (90) using eqns (A9) and (65). The functional (92) is derived from the substitution of eqn (A10) into the functional (91) and using eqns (70) and (74b, c). ■

Proposition 4

A kinematically admissible displacement field u^o_i is a (or the) solution of the problem P [eqns (1)–(9)] if and only if it minimizes (absolute minimum) the functional (88) [or any of the equivalent functionals (89)–(92)]. The problem P has at least one solution if and only if the functional (88) assumes, at the minimum, the following value:

$$F^0_{tpe} = \int_T \left\{ \frac{1}{2}\int_\Omega \sigma^{ef}_{ij}\varepsilon^{ef}_{ij}\,\mathrm{d}\Omega - \int_\Omega F_i u^{ef}_i\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i u^{ef}_i\,\mathrm{d}\Gamma \right\}\mathrm{d}t. \tag{93}$$

Proof. The proof is given using the functional $F^{\mathrm{V}}_{tpe}[u^o_i]$ and showing that the difference $F^{\mathrm{V}}_{tpe} - F^0_{tpe}$ is always non-negative, i.e.

$$F^{\mathrm{V}}_{tpe} - F^0_{tpe} = \frac{1}{2}\int_T \int_\Omega \tilde{\sigma}^s_{ij}\tilde{\varepsilon}^s_{ij}\,\mathrm{d}\Omega\,\mathrm{d}t \geqslant 0. \tag{94}$$

This can be easily recognized because the second member of eqn (94) represents the elastic deformation energy of the solid due to the volume forces $\tilde{F}_i = -[\Psi_{ij}(\boldsymbol{\varepsilon}^o)]_{,j} - F_i$ and the surface forces $\tilde{p}_i = \Psi_{ij}(\boldsymbol{\varepsilon}^o)n_j - p_i$ on Γ_p.

The functional F^{V}_{tpe} attains its minimum value F^0_{tpe} if and only if $\tilde{F}_i = 0$ and $\tilde{p}_i = 0$, i.e. if and only if

$$[\Psi_{ij}(\boldsymbol{\varepsilon}^o)]_{,j} + F_i = 0 \quad \text{in}\,\Omega\times\mathrm{T} \tag{95}$$

$$\Psi_{ij}(\boldsymbol{\varepsilon}^o)n_j = p_i \quad \text{on}\,\Gamma_p\times\mathrm{T} \tag{96}$$

which represent the equilibrium equations and the constitutive law. Equations (95) and (96), together with conditions (33) and (34), represent the whole of the governing equations of the original problem P [eqns (1)–(9)]. If, at the minimum

$$F_{tpe}^{V} > F_{tpe}^{0} \tag{97}$$

then no displacement field u_i^o [satisfying restrictions (33) and (34)] exists such as to satisfy both the equations (95) and (96) at the same time. Then the original problem P has no solution. ■

6. A PARTICULAR CLASS OF FUNCTIONALS

6.1. *Splitting of the extended complementary energy functional*

The functional (79) can be written as a sum of two partial functionals in the following form

$$F_{ce}^{I}[\sigma_{ij}^{*}] = F_{ce}^{Ia}[\sigma_{ij}^{*}] + F_{ce}^{Ib}[\sigma_{ij}^{*}], \tag{98}$$

where

$$F_{ce}^{Ia}[\sigma_{ij}^{*}] = \int_T \left\{ \frac{1}{2}\int_\Omega \sigma_{ij}^{*}\Phi_{ij}(\boldsymbol{\sigma}^{*})\,\mathrm{d}\Omega - \int_{\Gamma_u} \sigma_{ij}^{*}n_j v_i\,\mathrm{d}\Gamma \right\}\mathrm{d}t \tag{99}$$

$$F_{ce}^{Ib}[\sigma_{ij}^{*}] = \int_T \left\{ \frac{1}{2}\int_\Omega \sigma_{ij}^{d}B_{ijhk}\sigma_{hk}^{d}\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \Phi_{ij}^{n}(\boldsymbol{\sigma}^{*})\sigma_{ij}^{*}\,\mathrm{d}\Omega - \int_\Omega \Phi_{ij}^{n}(\boldsymbol{\sigma}^{*})\sigma_{ij}^{ef}\,\mathrm{d}\Omega \right\}\mathrm{d}t. \tag{100}$$

The following proposition may be proved.

Proposition 5

A statically admissible stress field σ_{ij}^{*} is a (or the) solution of the problem (1)–(9) only if (necessary condition) it minimizes (absolute minimum) both the functionals (99) and (100) and if (sufficient condition) for any couple $\sigma_{ij}^{*(1)}$ and $\sigma_{ij}^{*(2)}$ of admissible stresses the following condition holds everywhere in Ω and at any $t \in T$:

$$\sigma_{ij}^{*(1)}\Phi_{ij}^{n}(\boldsymbol{\sigma}^{*(1)}) + \sigma_{ij}^{*(2)}\Phi_{ij}^{n}(\boldsymbol{\sigma}^{*(2)}) - 2\sigma_{ij}^{*(1)}\Phi_{ij}^{n}(\boldsymbol{\sigma}^{*(2)}) \geqslant 0. \tag{101}$$

Proof. Let us consider the differences (where σ_{ij} is the actual stress distribution):

$$\Delta F_{ce}^{Ia} = F_{ce}^{Ia}[\sigma_{ij}^{*}] - F_{ce}^{Ia}[\sigma_{ij}] \tag{102}$$

$$\Delta F_{ce}^{Ib} = F_{ce}^{Ib}[\sigma_{ij}^{*}] - F_{ce}^{Ib}[\sigma_{ij}] \tag{103}$$

$$\Delta F_{ce}^{I} = F_{ce}^{I}[\sigma_{ij}^{*}] - F_{ce}^{I}[\sigma_{ij}] = \Delta F_{ce}^{Ia} + \Delta F_{ce}^{Ib} \tag{104}$$

and

$$\Delta\sigma_{ij} = \sigma_{ij}^{*} - \sigma_{ij} \tag{105}$$

$$\Delta\sigma_{ij}^{d} = \sigma_{ij}^{d}(\boldsymbol{\sigma}^{*}) - \sigma_{ij}^{d}(\boldsymbol{\sigma}). \tag{106}$$

The difference (102), using eqn (105) may be written as:

$$\Delta F_{ce}^{\mathrm{Ia}} = \int_T \left\{ \frac{1}{2} \int_\Omega \Delta\sigma_{ij} B_{ijhk} \Delta\sigma_{hk} \, \mathrm{d}\Omega + \int_\Omega \Delta\sigma_{ij} (B_{ijhk}\sigma_{hk} + \Phi_{ij}^n(\boldsymbol{\sigma})) \, \mathrm{d}\Omega \right.$$

$$\left. - \int_{\Gamma_u} \Delta\sigma_{ij} n_j v_i \, \mathrm{d}\Gamma + \frac{1}{2} \int_\Omega \sigma_{ij}^* \Phi_{ij}^n(\boldsymbol{\sigma}^*) \, \mathrm{d}\Omega + \frac{1}{2} \int_\Omega \sigma_{ij} \Phi_{ij}^n(\boldsymbol{\sigma}) \, \mathrm{d}\Omega - \int_\Omega \sigma_{ij}^* \Phi_{ij}^n(\boldsymbol{\sigma}) \, \mathrm{d}\Omega \right\} \mathrm{d}t, \quad (107)$$

where the second and third term of the second member vanish by virtue of the principle of virtual work. Using inequality (101) and assuming $\boldsymbol{\sigma}^{*(1)} = \boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}^{*(2)} = \boldsymbol{\sigma}$, the following may then be written:

$$\Delta F_{ce}^{\mathrm{Ia}} \geq 0 \quad (108)$$

$$\Delta F_{ce}^{\mathrm{Ia}} = 0 \quad \text{if and only if} \quad \sigma_{ij}^* = \sigma_{ij}. \quad (109)$$

This means that any (or the) solution of the original problem (1)–(9) minimizes the functional (99) (necessary condition) under the admissibility conditions of σ_{ij}^*. Similarly it is easy to prove that

$$\Delta F_{ce}^{\mathrm{Ib}} \geq 0 \quad (110)$$

$$\Delta F_{ce}^{\mathrm{Ib}} = 0 \quad \text{if and only if} \quad \sigma_{ij}^* = \sigma_{ij}. \quad (111)$$

Then a (or the) solution of the original problem (1)–(9) minimizes the functional (100) (necessary conditions) under the admissibility conditions of σ_{ij}^*. ■

6.2. *Splitting of the extended total potential energy functional*

In the same way as shown in Section 6.1 the functional (88) may be written as a sum of two other functionals in the following form

$$F_{tpe}^{\mathrm{I}}[u_i^o] = F_{tpe}^{\mathrm{Ia}}[u_i^o] + F_{tpe}^{\mathrm{Ib}}[u_i^o], \quad (112)$$

where

$$F_{tpe}^{\mathrm{Ia}}[u_i^o] = \int_T \left\{ \frac{1}{2} \int_\Omega \varepsilon_{ij}^o \Psi_{ij}(\varepsilon^o) \, \mathrm{d}\Omega - \int_\Omega F_i u_i^o \, \mathrm{d}\Omega - \int_{\Gamma_p} p_i u_i^o \, \mathrm{d}\Gamma \right\} \mathrm{d}t \quad (113)$$

$$F_{tpe}^{\mathrm{Ib}}[u_i^o] = \int_T \left\{ \frac{1}{2} \int_\Omega \varepsilon_{ij}^s D_{ijhk} \varepsilon_{hk}^s \, \mathrm{d}\Omega + \frac{1}{2} \int_\Omega \Psi_{ij}^n(\boldsymbol{\varepsilon}^o) \varepsilon_{ij}^o \, \mathrm{d}\Omega - \int_\Omega \Psi_{ij}^n(\boldsymbol{\varepsilon}^o) \varepsilon_{ij}^{ef} \, \mathrm{d}\Omega \right\} \mathrm{d}t. \quad (114)$$

The following proposition may be proved.

Proposition 6

A kinematically admissible displacement field u_i^o is a (or the) solution of the problem (1)–(9) only if (necessary condition) it minimizes (absolute minimum) both the functionals (113) and (114) and if (sufficient condition) for any couple $u_i^{o(1)}$ and $u_i^{o(2)}$ of admissible displacements the following condition holds everywhere in Ω and at any $t \in T$:

$$\varepsilon_{ij}^{o(1)} \Psi_{ij}^n(\varepsilon^{o(1)}) + \varepsilon_{ij}^{o(2)} \Psi_{ij}^n(\varepsilon^{o(2)}) - 2\varepsilon_{ij}^{o(1)} \Psi_{ij}^n(\varepsilon^{o(2)}) \geq 0. \quad (115)$$

No proof is given here because it is analogous to that given in Section 6.1.

Remark. In the presence of a material behaviour complying with inequalities (101) and (115) for every couple of stresses $\sigma_{ij}^{(1)}, \sigma_{ij}^{(2)}$ (even non-statically admissible) and strains $\varepsilon_{ij}^{(1)}, \varepsilon_{ij}^{(2)}$ (even non-kinematically admissible), respectively, and in the presence of an invertible constitutive law, the relations (101) and (115) are equivalent to each other, as is easily shown in the following. Let us write relation (101) in the new form:

$$\sigma_{ij}^{(1)}\Phi_{ij}(\boldsymbol{\sigma}^{(1)})+\sigma_{ij}^{(2)}\Phi_{ij}(\boldsymbol{\sigma}^{(2)})-2\sigma_{ij}^{(1)}\Phi_{ij}(\boldsymbol{\sigma}^{(2)})-(\sigma_{ij}^{(1)}-\sigma_{ij}^{(2)})B_{ijhk}(\sigma_{hk}^{(1)}-\sigma_{hk}^{(2)}) \geqslant 0 \tag{116}$$

which transforms [taking into account the reversibility relation (10)] into:

$$\varepsilon_{ij}^{(1)}\Psi_{ij}(\boldsymbol{\varepsilon}^{(1)})+\varepsilon_{ij}^{(2)}\Psi_{ij}(\boldsymbol{\varepsilon}^{(2)})-2\varepsilon_{ij}^{(1)}\Psi_{ij}(\boldsymbol{\varepsilon}^{(2)})-(\sigma_{ij}^{(1)}-\sigma_{ij}^{(2)})B_{ijhk}(\sigma_{hk}^{(1)}-\sigma_{hk}^{(2)}) \geqslant 0 \tag{117}$$

or in the following equivalent form:

$$\varepsilon_{ij}^{(1)}\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{(1)})+\varepsilon_{ij}^{(2)}\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{(2)})-2\varepsilon_{ij}^{(1)}\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{(2)})-[\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{(1)})-\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{(2)})]B_{ijhk}[\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{(1)})-\Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{(2)})] \geqslant 0. \tag{118}$$

The positive definiteness of the elastic modulus tensor D_{ijhk} [see eqn (14)] [and then of the non-negativeness of last term of eqn (118)] implies:

$$\varepsilon_{ij}^{(1)}\Psi_{ij}^{n}(\varepsilon^{(1)})+\varepsilon_{ij}^{(2)}\Psi_{ij}^{n}(\varepsilon^{(2)})-2\varepsilon_{ij}^{(1)}\Psi_{ij}^{n}(\varepsilon^{(2)}) \geqslant 0. \tag{119}$$

In a similar way it is possible to derive relation (101) from (115), under the same hypotheses.

7. LINKS WITH SOME CLASSICAL FUNCTIONALS

Reference is made in the following to the classical principles of the linear elasticity and incremental plasticity: this implies that in the next subsections, the time dependence of the material behaviour and of the problem will be ignored.

7.1. *Linear elasticity energy functionals*

It is easy to show that in the linear elasticity case, as a consequence of the absence of the non-linear part $\Phi_{ij}^{n}(\cdot)$ of the constitutive law (12), and $\Psi_{ij}^{n}(\cdot)$ of the constitutive law (11) the functionals (79) and (88) transform into the classical ones of the complementary and total potential energy:

$$F_{ce}[\sigma_{ij}^{*}] = \frac{1}{2}\int_{\Omega}\sigma_{ij}^{*}B_{ijhk}\sigma_{hk}^{*}\,\mathrm{d}\Omega-\int_{\Gamma_u}\sigma_{ij}^{*}n_j v_i\,\mathrm{d}\Gamma \tag{120}$$

$$F_{tp}[u_i^{o}] = \frac{1}{2}\int_{\Omega}\varepsilon_{ij}^{o}D_{ijhk}\varepsilon_{hk}^{o}\,\mathrm{d}\Omega-\int_{\Omega}F_i u_i^{o}\,\mathrm{d}\Omega-\int_{\Gamma_p}p_i u_i^{o}\,\mathrm{d}\Gamma. \tag{121}$$

When $\Phi_{ij}^{n}(\boldsymbol{\sigma}^{*})$ is considered as a known distortion θ_{ij} (as in the case of a given thermal type distortion) then, apart from non-essential constant terms, the functional (79) transforms into the following well-known Colonnetti's functional (Castigliano, 1879; Colonnetti, 1918, 1950; Reissner, 1931):

$$F_{c}[\sigma_{ij}^{*}] = \frac{1}{2}\int_{\Omega}\sigma_{ij}^{*}B_{ijhk}\sigma_{hk}^{*}\,\mathrm{d}\Omega-\int_{\Gamma_u}\sigma_{ij}^{*}n_j v_i\,\mathrm{d}\Gamma+\int_{\Omega}\theta_{ij}\sigma_{ij}^{*}\,\mathrm{d}\Omega. \tag{122}$$

In the same way, when $B_{ijhk}\Psi_{hk}^{n}(\boldsymbol{\varepsilon}^{o})$ is considered as a known distortion θ_{ij}, apart from non-essential constant terms, the functional (88) transforms into the following Greenberg's functional (Greenberg, 1949a):

$$F_{tpe}[u_i^o] = \frac{1}{2}\int_\Omega \varepsilon_{ij}^o D_{ijhk}\varepsilon_{hk}^o \,d\Omega - \int_\Omega F_i u_i^o \,d\Omega - \int_{\Gamma_f} p_i u_i^o \,d\Gamma + \int_\Omega \theta_{ij} D_{ijhk}\varepsilon_{hk}^o \,d\Omega. \tag{123}$$

7.2. *Incremental plasticity functionals*

When dealing with incremental elastoplasticity, the sufficient condition (101) for which the two partial functionals F_{ce}^{Ia}, F_{ce}^{Ib} of F_{ce}^{I} attain the minimum at the solution, becomes (where a superimposed point of the symbol means infinitesimal increment with respect to t):

$$\dot\sigma_{ij}^{*(1)}\dot\varepsilon_{ij}^{*(1)p} + \dot\sigma_{ij}^{*(2)}\dot\varepsilon_{ij}^{*(2)p} - 2\dot\sigma_{ij}^{*(1)}\dot\varepsilon_{ij}^{*(2)p} \geq 0 \tag{124}$$

which may be easily recognized as one of the fundamental consequences of Drucker's stability postulate (Drucker, 1951), the plastic incremental strains $\dot\varepsilon_{ij}^{*(1)p}, \dot\varepsilon_{ij}^{*(2)p}$ (corresponding to the incremental stress $\dot\sigma_{ij}^{*(1)}, \dot\sigma_{ij}^{*(2)}$ starting from the same stress state $\bar\sigma_{ij}$ on the yield surface) being given by:

$$\dot\varepsilon_{ij}^{*(1)p} = \Phi_{ij}^{p}(\dot\sigma_{ij}^{*(1)}), \quad \dot\varepsilon_{ij}^{*(2)p} = \Phi_{ij}^{p}(\dot\sigma_{ij}^{*(2)}). \tag{125}$$

Then the two partial functional F_{ce}^{Ia} [eqn (99)], F_{ce}^{Ib} [eqn (100)], written in incremental form, become:

$$F_{ce}^{Ia}[\dot\sigma_{ij}^*] = \frac{1}{2}\int_\Omega \dot\sigma_{ij}^*\dot\varepsilon_{ij}^*(\dot{\boldsymbol\sigma}^*)\,d\Omega - \int_{\Gamma_u} \dot\sigma_{ij}^* n_j \dot v_i \,d\Gamma \tag{126}$$

$$F_{ce}^{Ib}[\dot\sigma_{ij}^*] = \frac{1}{2}\int_\Omega \dot\sigma_{ij}^d B_{ijhk}\dot\sigma_{hk}^d \,d\Omega + \frac{1}{2}\int_\Omega \dot\varepsilon_{ij}^{*p}(\dot{\boldsymbol\sigma}^*)\dot\sigma_{ij}^* \,d\Omega - \int_\Omega \dot\varepsilon_{ij}^{*p}(\dot{\boldsymbol\sigma}^*)\dot\sigma_{ij}^d \,d\Omega. \tag{127}$$

It is easy to recognize in eqn (126) the Prager–Hodge functional (Prager, 1942, 1946; Hodge and Prager, 1948), while functional (127) seems to be new, to the author's knowledge.

In the same way the two partial functionals F_{tpe}^{Ia} [eqn (113)], F_{tpe}^{Ib} [eqn (114)] of F_{tpe}^{I} become:

$$F_{tpe}^{Ia}[\dot u_i^o] = \frac{1}{2}\int_\Omega \dot\varepsilon_{ij}^o\dot\sigma_{ij}^o(\dot\varepsilon^o)\,d\Omega - \int_\Omega \dot F_i \dot u_i^o \,d\Omega - \int_{\Gamma_p} \dot p_i \dot u_i^o \,d\Gamma \tag{128}$$

$$F_{tpe}^{Ib}[\dot u_i^o] = \frac{1}{2}\int_\Omega \dot\varepsilon_{ij}^o D_{ijhk}\dot\varepsilon_{hk}^o \,d\Omega + \frac{1}{2}\int_\Omega \dot\sigma_{ij}^{o^p}(\dot\varepsilon^o)\dot\varepsilon_{ij}^o \,d\Omega - \int_\Omega \dot\sigma_{ij}^{o^p}(\dot{\boldsymbol\varepsilon}^o)\dot\varepsilon_{ij}^{o^t} \,d\Omega, \tag{129}$$

where

$$\dot\sigma_{ij}^{p} = \Psi_{ij}^{p}(\dot\varepsilon^o) = -D_{ijhk}\dot\varepsilon_{hk}^{o^p}. \tag{130}$$

It is easy to recognize in eqn (128) the Greenberg–Prager functional (Greenberg, 1949b; Prager, 1942, 1946) while functional (129) seems to be new, to the author's knowledge.

Connections of the above functionals (126)–(129) with other more recent incremental plasticity functionals (Ceradini, 1965, 1966; Capurso, 1969; Maier, 1968, 1969, 1970; Capurso and Maier, 1970) can be easily recognized when the incremental elastoplastic constitutive law is written in terms of plastic multiplier rates.

8. SIMPLE ILLUSTRATIVE APPLICATIONS

8.1. *Elastic spring with hardening or softening*

The aim of the example is to show the use of Colonnetti's extended minimum principle in the dual form [eqn (88)] when the problem has more than one solution or when no solution exists.

In the example of Fig. 1, the spring has a two-branch piecewise-linear elastic behaviour with the slope of the last branch given by $\alpha k(k > 0)$ with $-\infty \leqslant \alpha \leqslant 1$ [see Fig. 1(a)]. The constitutive law may be written in the form

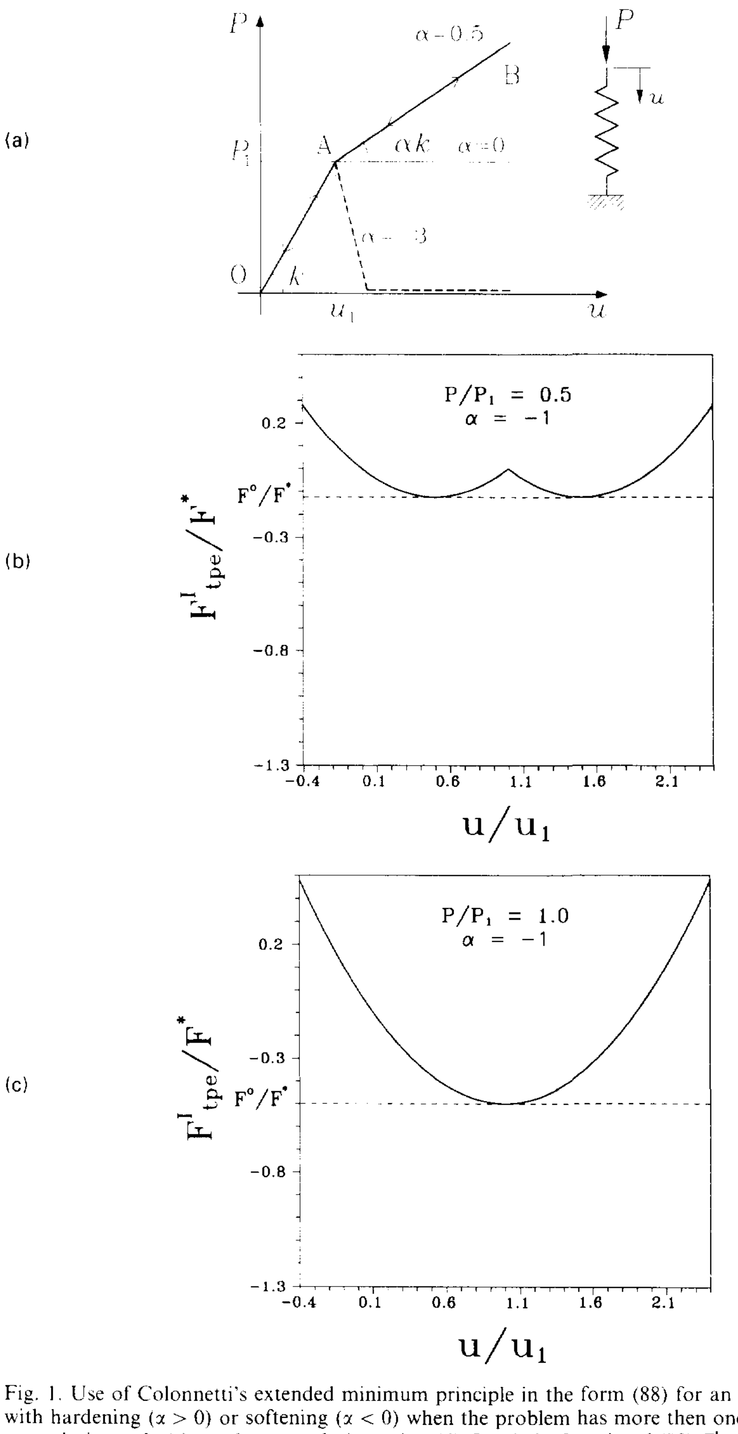
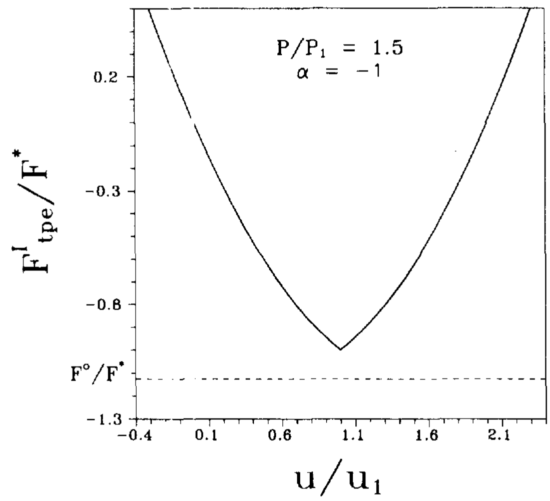


Fig. 1. Use of Colonnetti's extended minimum principle in the form (88) for an elastic spring (a) with hardening ($\alpha > 0$) or softening ($\alpha < 0$) when the problem has more then one solution (b), or one solution only (c) or when no solution exists (d). In (e) the functional (88) F^{I}_{tpe} (u, P) is depicted as a surface in the space $F^{\mathrm{I}}_{tpe}-P-u$. In Figs 1(b)–(e) $F^* = P_1^2/k$. (*Continued opposite.*)

(d)



(e)

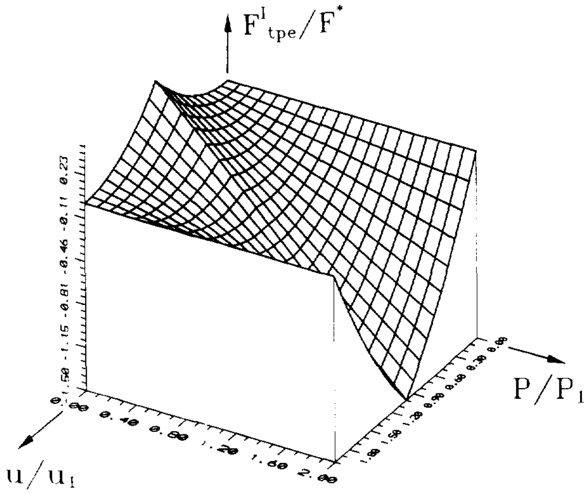


Fig. 1—*Continued.*

$$P = ku + ck(\alpha-1)(u-u_1) \tag{131}$$

with $c = 0$ for $u \leqslant u_1$ and $c = 1$ for $u > u_1$. Then the functional (88) becomes (for $u < u_1 - (P_1/\alpha k)$ when $\alpha < 0$):

$$F^1_{tpe}[u] = \tfrac{1}{2}ku^2 - Pu + ck(\alpha-1)(u-u_1)\left(u - \frac{P}{k}\right) + \tfrac{1}{2}ck(\alpha-1)^2(u-u_1)^2 \tag{132}$$

and its minimum value F^0_{tpe} to be reached for the existence of the problem solution, becomes:

$$F^0_{tpe} = -\frac{1}{2}\frac{P^2}{k}. \tag{133}$$

In Figs 1(b–d) the plots $F^1_{tpe}[u]$ for $\alpha = -1$ and for different values of the applied load are given. From Figs 1(b, c) it may be deduced that the solution exists for the given value of the load, since a minimum of the functional assumes the value F^0_{tpe}; from Fig. 1(d), instead it appears that the problem has no solution because the minimum of the functional is greater than F^0_{tpe}. The shape of the surface $F^1_{tpe}[u, P]$ is shown in Fig. 1(e).

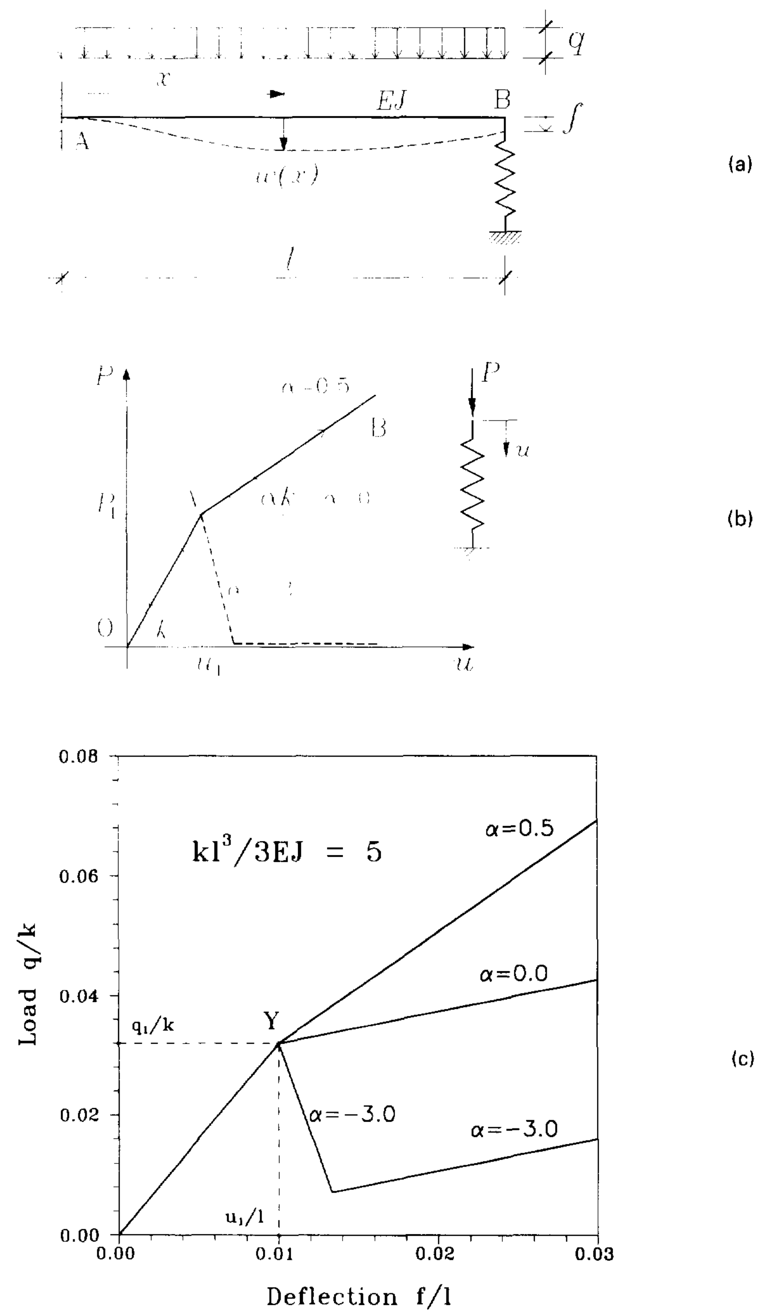


Fig. 2. Use of Colonnetti's extended minimum principle in the form (88) for the cantilever of (a) with end spring behaviour given in (b). (c) gives the supported end deflexion f for any given load q for different values of the hardening coefficient (hardening for $\alpha > 0$, softening for $\alpha < 0$).

8.2. *Propped cantilever supported on an elastic spring with hardening or softening*

The aim of this example is to show the construction of Colonnetti's extended functional in the dual form [eqn (88)] in the case of a very simple structure.

In the example of Fig. 2 the linear elastic cantilever of Fig. 2(a) (with constant bending stiffness EJ) is simply supported at B with a spring of the type of Fig. 1. In this case the solution exists for every value of the distributed load $q(x)$ on the cantilever and for any value (even negative) of the α parameter. The functional F^{I}_{tpe} [eqn (88)] in the unknown kinematically admissible displacement function $w(x)$, assumes the form:

$$F^{\mathrm{I}}_{tpe}[w] = \frac{1}{2}\int_0^l EJ\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + \tfrac{1}{2}kf^2 - \int_0^l q(x)w(x)\,\mathrm{d}x$$

$$+\Psi''(f)\left(f - \frac{3ql^4}{8(kl^3+3EJ)}\right) + \tfrac{1}{2}\Psi''(f)\frac{l^3}{kl^3+3EJ}\Psi''(f), \quad (134)$$

where

$$\left(\text{for} \quad f < u_1 - \frac{P_1}{\alpha k} \quad \text{when} \quad \alpha < 0\right)$$

$$\Psi'(f) = ck(\alpha - 1)(f - u_1). \quad (135)$$

with $c = 0$ for $u \leqslant u_1$ and $c = 1$ for $u > u_1$. In Fig. 2(c) the plot of the applied load vs the end deflection of section B of the cantilever for different values of α is given. In particular, for $\alpha = -3$ and for $P < P_1$, three solutions exist.

8.3. *Deflections of a non-homogeneous viscoelastic beam*

The aim of this example is to show the application of Colonnetti's extended functional in the dual form [eqn (91)] in the case of time-dependent constitutive law.

Consider the non-homogeneous linear viscoelastic beam of Fig. 3(a) doubly clamped (in A and B) and with a concentrated known load P at the centre line C. The problem is to find the displacement function $w(x;t)$ for $0 \leqslant t \leqslant T$. The beam left side is elastic with Young's modulus E_1, while its right side is viscoelastic with the following moment–curvature relationship:

$$M(x;t) = r(0)J\frac{\partial w(x;t)}{\partial x} - J\int_0^t \frac{\partial r(t-\tau)}{\partial \tau}\frac{\partial w(x,\tau)}{\partial x}\mathrm{d}\tau, \quad (136)$$

where $r(t)$ is a three-parameter viscoelastic hereditary Kelvin–Voigt relaxation function:

$$r(t) = E_\infty - (E_1 - E_\infty)\mathrm{e}^{-t/T^*}, \quad (137)$$

where

$$E_\infty = \frac{E_1E_2}{E_1+E_2}, \quad T^* = \frac{\eta}{E_1+E_2}, \quad (138)$$

E_1, E_2, η being the parameters of the Kelvin–Voigt rheological model of Fig. 3(b). The functional F^{IV}_{tpe} [eqn (91)], in the unknown kinematically admissible displacement function $w(x;t)$, assumes the form:

$$F^{\mathrm{IV}}_{tpe}[w(x;t)] = \frac{1}{2}\int_0^T\int_0^l E_1J\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x\,\mathrm{d}t - \int_0^T P\hat{w}(l/2;t)\,\mathrm{d}t, \quad (139)$$

where $\hat{w}$ is the solution of the elastic auxiliary problem, i.e. of the elastic problem with elastic bending stiffness E_1J for the entire beam, under the transverse load:

$$\hat{q}(x;t) = Jr(0)\frac{\partial^4 w(x;t)}{\partial x^4} - J\int_0^t \frac{\partial r(t-\tau)}{\partial \tau}\frac{\partial^4 w(x,\tau)}{\partial x^4}\mathrm{d}\tau \quad (140)$$

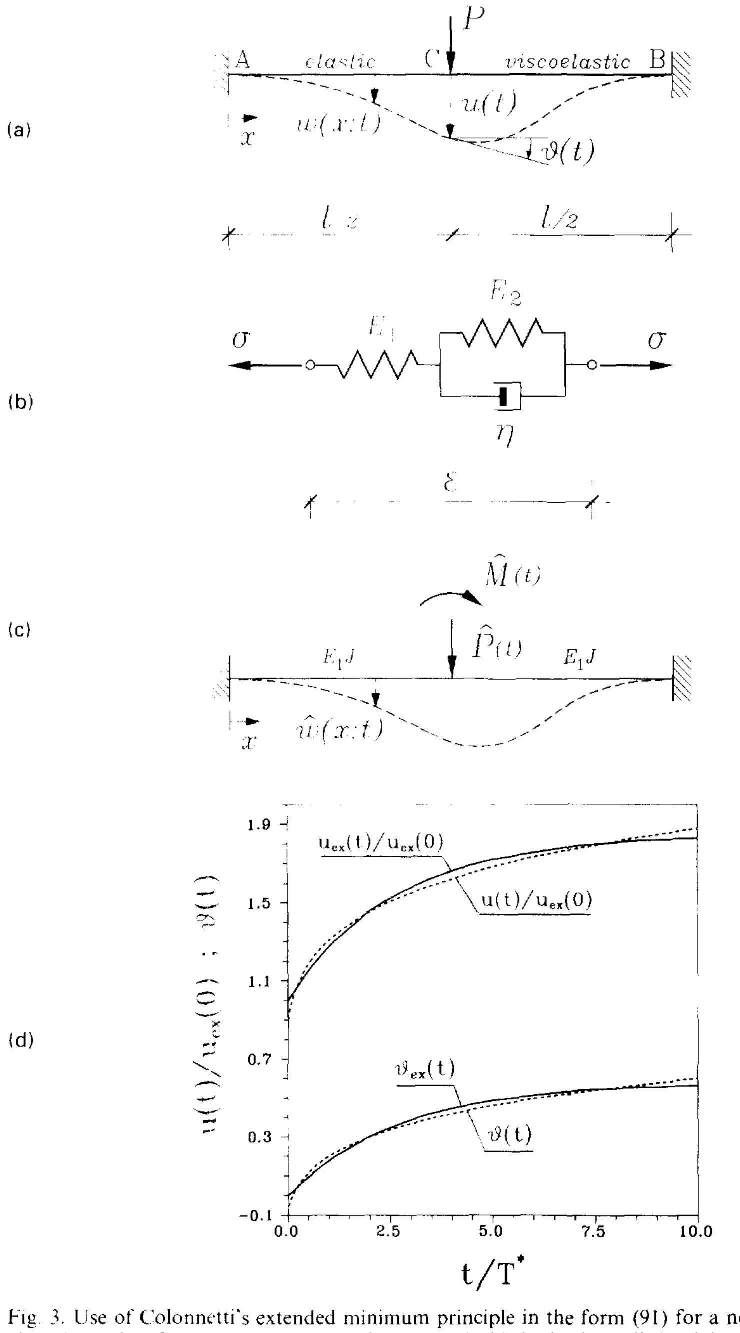


Fig. 3. Use of Colonnetti's extended minimum principle in the form (91) for a non-homogeneous viscoelastic doubly clamped beam (a). The beam left side is elastic, while its right side is viscoelastic with the three parameter rheological Kelvin–Voigt model of (b). (c) shows the particular auxiliary loads deriving from the chosen displacement compatible cubic function $w(x)$, used for the determination of the elastic auxiliary problem solution $\hat{w}(x)$. In (d) is shown a comparison between the exact solution and the approximate solution with three temporal degrees of freedom for the displacement $u(t)$ and the rotation $\theta(t)$ at the centre line C.

with $r(t) \equiv E_1 =$ constant for the beam left side. It is possible to represent the unknown function $w(x;t)$ with a displacement compatible cubic function of x separately over the left side and right side of the beam, i.e.

$$w(x;t) = N_1(x)u(t) + N_2(x)l\theta(t) \quad \text{for } 0 \leqslant x \leqslant l/2 \tag{141}$$

$$w(x;t) = N_3(x)u(t) + N_4(x)l\theta(t) \quad \text{for } l/2 \leqslant x \leqslant l \tag{142}$$

with

$$N_1(x) = 12\left(\frac{x}{l}\right)^2 - 16\left(\frac{x}{l}\right)^3$$

$$N_2(x) = -2\left(\frac{x}{l}\right)^2 + 4\left(\frac{x}{l}\right)^3$$

$$N_3(x) = -4 + 24\left(\frac{x}{l}\right) - 36\left(\frac{x}{l}\right)^2 + 16\left(\frac{x}{l}\right)^3$$

$$N_4(x) = -2 + 8\left(\frac{x}{l}\right) - 10\left(\frac{x}{l}\right)^2 + 4\left(\frac{x}{l}\right)^3, \quad (143)$$

where $u(t)$ and $\theta(t)$ are approximate with given time shape functions over the whole time interval $0 \leqslant t \leqslant T$, i.e.

$$u(t) = \mathbf{M}_u^T(t)\boldsymbol{\alpha} = [M_{u_1}, M_{u_2}, \ldots, M_{u_n}][\alpha_1, \ldots, \alpha_n]^T \quad (144)$$

$$\theta(t) = \mathbf{M}_\theta^T(t)\boldsymbol{\beta} = [M_{\theta_1}, M_{\theta_2}, \ldots, M_{\theta_n}][\beta_1, \ldots, \beta_n]^T. \quad (145)$$

Using eqns (141)–(145), the load $\hat{q}(x;\, t)$ of eqn (140) transforms into the following concentrated loads (concentrated force $\hat{P}$ and moment $\hat{M}$) at the centre line C [see Fig. 3(c)]:

$$\hat{P} = \frac{E_1 J}{l^3}\{12[(M_{u_1}(t_0) + U_{u_1}(t) + V_{u_1}(t))\alpha_1 + \cdots + (M_{u_n}(t_0) + U_{u_n}(t) + V_{u_n}(t))\alpha_n]$$
$$-6l[(M_{\theta_1}(t_0) - U_{\theta_1}(t) - V_{\theta_1}(t))\beta_1 + \cdots + (M_{\theta_n}(t_0) - U_{\theta_n}(t) - V_{\theta_n}(t))\beta_n]\} \quad (146)$$

$$\hat{M} = \frac{E_1 J}{l^3}\{-6l[(M_{u_1}(t_0) - U_{u_1}(t) - V_{u_1}(t))\alpha_1 + \cdots + (M_{u_n}(t_0) - U_{u_n}(t) - V_{u_n}(t))\alpha_n]$$
$$+4l^2[(M_{\theta_1}(t_0) + U_{\theta_1}(t) + V_{\theta_1}(t))\beta_1 + \cdots + (M_{\theta_n}(t_0) + U_{\theta_n}(t) + V_{\theta_n}(t))\beta_n]\}, \quad (147)$$

where

$$U_{u_i}(t) = \frac{r(t, t_0)}{E_1} M_{u_i}(t_0); \quad U_{\theta_i}(t) = \frac{r(t, t_0)}{E_1} M_{\theta_i}(t_0) \quad \text{for } 1 \leqslant i \leqslant n \quad (148)$$

$$V_{u_i}(t) = \int_0^t \frac{r(t, \tau)}{r(t, t_0)} \frac{\mathrm{d}M_{u_i}}{\mathrm{d}\tau}\mathrm{d}\tau; \quad V_{\theta_i}(t) = \int_0^t \frac{r(t, \tau)}{r(t, t_0)} \frac{\mathrm{d}M_{\theta_i}}{\mathrm{d}\tau}\mathrm{d}\tau \quad \text{for } 1 \leqslant i \leqslant n. \quad (149)$$

Now the solution $\hat{w}(x;\, t)$ of the elastic auxiliary problem becomes:

$$\hat{w}(x;\, t) = N_1(x)\frac{l^3}{24E_1 J}\hat{P}(t) + N_2(x)\frac{l^2}{8E_1 J}\hat{M}(t) \quad \text{for } 0 \leqslant x \leqslant l/2 \quad (150)$$

$$\hat{w}(x;\, t) = N_3(x)\frac{l^3}{24E_1 J}\hat{P}(t) + N_4(x)\frac{l^2}{8E_1 J}\hat{M}(t) \quad \text{for } l/2 \leqslant x \leqslant l. \quad (151)$$

Substituting eqns (150) and (151) in (139) we obtain a $2n$ parameter quadratic function whose stationarity leads to a linear system of $2n$ equations. In Fig. 3(d) the exact viscoelastic solution $u_{ex}(t)$ and $\theta_{ex}(t)$ of the problem of Fig. 3(a) is compared with the three parameter

($n = 3$) solution in the time interval ($0 \leqslant t/T^* \leqslant 10$), having used the following time logarithmic shape functions:

$$\begin{aligned} M_{u_1} &= M_{\sigma_1} = 1 \\ M_{u_2} &= M_{\sigma_2} = \ln(t+1) \\ M_{u_3} &= M_{\sigma_3} = (\ln(t+1))^2. \end{aligned} \tag{152}$$

The diagram of Fig. 3(d) shows that the present method allows a good interpretation of the viscoelastic behaviour of the beam.

9. CONCLUSIONS

Under the assumptions of general non-linear invertible constitutive law with the form (11), (12), two minimum principles [eqns (79) and (88)] have been derived as a generalization of Colonnetti's principle. The above result was obtained despite the absence of a potential of the problem (which excludes the existence of extremal formulations in the classical sense), by introducing the notion of *elastic auxiliary problem.*

Main characterizations of the proposed generalization of Colonnetti's principle (which consider the material non-linearities such as distortions developing in the continuum assumed in linear elastic regime) are:

(1) the energy meaning of the functionals (which is useful for their construction);

(2) the linear elasticity and incremental elastoplasticity classical formulations may be found as particular cases of the generalized formulations;

(3) the formulations are valid in all cases even in the lack of existence or uniqueness of the solution. In fact when there is no uniqueness, all solutions can be determined, while we can easily see the lack of existence of the solution testing the extremal value of the functional.

It is worth noting that a relatively straightforward generalization of the present study should be to materials whose elastic part of the constitutive law is non-linear. Another possible generalization is to the case of large strains ("geometric" non-linearity).

From the computational point of view, the greatest drawback in the use of the functionals suggested here rests in the solution of the elastic auxiliary problem. However, this drawback could be overcome through the application of the boundary element technique. In this way the symmetric Galerkin double-integration approach seems to be particularly promising (Maier and Polizzotto, 1987; Polizzotto, 1988; Comi and Maier, 1992).

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APPENDIX

The following relations can be easily proved.

Using eqns (36) and (41) and twice the principle of virtual work we have

$$\int_\Omega \Phi''_{ij}(\boldsymbol{\sigma}^*)\sigma_{ij}^d\,\mathrm{d}\Omega = -\int_\Omega \sigma_{ij}^d\varepsilon_{ij}^d\,\mathrm{d}\Omega + \int_\Omega \varepsilon_{ij}^d\sigma_{ij}^d\,\mathrm{d}\Omega = -\int_\Omega F_iu_i^d\,\mathrm{d}\Omega - \int_{\Gamma_p} p_iu_i^d\,\mathrm{d}\Gamma + \int_{\Gamma_u} v_in_j\sigma_{ij}^d\,\mathrm{d}\Gamma. \tag{A1}$$

Taking into account eqns (36) and (41) the following can be written

$$\frac{1}{2}\int_\Omega \sigma_{ij}^dB_{ijhk}\sigma_{hk}^d\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega \Phi''_{ij}(\boldsymbol{\sigma}^*)\sigma_{ij}^d\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \varepsilon_{ij}^d\sigma_{ij}^d\,\mathrm{d}\Omega. \tag{A2}$$

where the last term of the second member vanishes due to the principle of virtual work.

Using the principle of virtual work and eqn (41) the following relation holds

$$\frac{1}{2}\int_\Omega \Phi''_{ij}(\boldsymbol{\sigma}^*)\sigma_{ij}^*\,\mathrm{d}\Omega - \frac{1}{2}\int_\Omega F_iu_i^d\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Gamma_p} p_iu_i^d\,\mathrm{d}\Gamma = \frac{1}{2}\int_\Omega \Phi''_{ij}(\boldsymbol{\sigma}^*)\sigma_{ij}^*\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \sigma_{ij}^*\varepsilon_{ij}^d\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega \sigma_{ij}^*B_{ijhk}\sigma_{hk}^d\,\mathrm{d}\Omega. \tag{A3}$$

Due to the principle of virtual work

$$\frac{1}{2}\int_\Omega \varepsilon_{ij}^d\sigma_{ij}^d\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega F_iu_i^d\,\mathrm{d}\Omega - \frac{1}{2}\int_{\Gamma_p} p_iu_i^d\,\mathrm{d}\Gamma = 0. \tag{A4}$$

Using the simmetry of the elastic constitutive law and using the principle of virtual work twice, the following can be written:

$$\frac{1}{2}\int_{\Omega} \hat{\varepsilon}_{ij}^{d}\sigma_{ij}^{ef}\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Omega} \varepsilon_{ij}^{ef}\hat{\sigma}_{ij}^{d}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \hat{\sigma}_{ij}^{d}\,\mathrm{d}\Gamma = \int_{\Omega} \varepsilon_{ij}^{ef}\hat{\sigma}_{ij}^{d}\,\mathrm{d}\Omega - \int_{\Gamma_u} v_i n_j \hat{\sigma}_{ij}^{d}\,\mathrm{d}\Gamma = \int_{\Omega} F_i u_i^{ef}\,\mathrm{d}\Omega + \int_{\Gamma_p} p_i u_i^{ef}\,\mathrm{d}\Gamma$$

$$= \int_{\Omega} \sigma_{ij}^{ef}\varepsilon_{ij}^{ef}\,\mathrm{d}\Omega - \int_{\Gamma_u} \sigma_{ij}^{ef} n_j v_i\,\mathrm{d}\Gamma. \quad \text{(A5)}$$

In the same way the following relations hold:

$$\int_{\Omega} \Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{o})\varepsilon_{ij}^{ef}\,\mathrm{d}\Omega = \int_{\Omega} \sigma_{ij}^{ef}\varepsilon_{ij}^{s}\,\mathrm{d}\Omega - \int_{\Omega} \varepsilon_{ij}^{ef}\sigma_{ij}^{s}\,\mathrm{d}\Omega = \int_{\Omega} F_i u_i^{s}\,\mathrm{d}\Omega + \int_{\Gamma_p} p_i u_i^{s}\,\mathrm{d}\Gamma - \int_{\Gamma_u} v_i n_j \sigma_{ij}^{s}\,\mathrm{d}\Gamma; \quad \text{(A6)}$$

$$\frac{1}{2}\int_{\Omega} \varepsilon_{ij}^{s} D_{ijhk}\varepsilon_{hk}^{s}\,\mathrm{d}\Omega = \frac{1}{2}\int_{\Omega} \Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{o})\varepsilon_{ij}^{s}\,\mathrm{d}\Omega; \quad \text{(A7)}$$

$$\frac{1}{2}\int_{\Omega} \Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{o})\varepsilon_{ij}^{o}\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Gamma_u} \sigma_{ij}^{s} n_j v_i\,\mathrm{d}\Gamma = \frac{1}{2}\int_{\Omega} \Psi_{ij}^{n}(\boldsymbol{\varepsilon}^{o})\varepsilon_{ij}^{o}\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Omega} \sigma_{ij}^{s}\varepsilon_{ij}^{o}\,\mathrm{d}\Omega = \frac{1}{2}\int_{\Omega} \varepsilon_{ij}^{o} D_{ijhk}\varepsilon_{hk}^{s}\,\mathrm{d}\Omega; \quad \text{(A8)}$$

$$\frac{1}{2}\int_{\Omega} \Psi_{ij}(\boldsymbol{\varepsilon}^{o})\hat{\varepsilon}_{ij}^{s}\,\mathrm{d}\Omega = \frac{1}{2}\int_{\Omega} \hat{\sigma}_{ij}^{s}\hat{\varepsilon}_{ij}^{s}\,\mathrm{d}\Omega - \frac{1}{2}\int_{\Gamma_u} \sigma_{ij}^{s} n_j v_i\,\mathrm{d}\Gamma; \quad \text{(A9)}$$

$$\frac{1}{2}\int_{\Omega} \hat{\sigma}_{ij}^{s}\varepsilon_{ij}^{ef}\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Omega} \sigma_{ij}^{ef}\hat{\varepsilon}_{ij}^{s}\,\mathrm{d}\Omega - \int_{\Omega} F_i \hat{u}_i^{s}\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i \hat{u}_i^{s}\,\mathrm{d}\Gamma = \int_{\Omega} \sigma_{ij}^{ef}\hat{\varepsilon}_{ij}^{s}\,\mathrm{d}\Omega - \int_{\Omega} F_i \hat{u}_i^{s}\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i \hat{u}_i^{s}\,\mathrm{d}\Gamma = \int_{\Gamma_u} \sigma_{ij}^{ef} n_j v_i\,\mathrm{d}\Gamma$$

$$= \int_{\Omega} \sigma_{ij}^{ef}\varepsilon_{ij}^{ef}\,\mathrm{d}\Omega - \int_{\Omega} F_i u_i^{ef}\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i u_i^{ef}\,\mathrm{d}\Gamma. \quad \text{(A10)}$$